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On Nearest Points of a Set

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ABSTRACT

In this paper, the main problem is concerned with the existence and uniqueness of nearest points of a given point in a set with weak *P*-property and generalised weak *P*- property.

***Key words*:** Proximinal set, *P*- property, weak *P*-property, generalised weak *P*- property
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1. INTRODUCTION

I Van Singer laid the foundation of Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces. [1] The problem of best approximation has a long history and gives rise to a lot of notions and techniques useful in functional analysis. In fact: Since 1970, when [1] has gone to print, the theory of best approximation in Banach spaces has developed rapidly and the number of papers in this field is growing continuously. There appeared several application of nearest point set. An application of distance sets to linear inequalities has been cited by Cheney and Goldstein. [14] P-property of a Banach space has been introduced by V. Sanker Raj and A. A. Eldred. [2] They have characterized different geometric properties of Banach space with the use of P-property. The P-property and its weak form called weak P-property are used for the existence and uniqueness of best proximity point and convergence of iterative sequence of various contraction mappings. [2,3,5-11] In this paper, we sketch some of the key results known about uniqueness and existence of nearest point in metric space and use the weak form of P-property called weak P-property and generalized weak P- property for the existence and uniqueness of nearest points of a given point in a set.

Definition 1.1. Let M be a nonempty set in a metric space, (X, d) and for any $x\in X \setminus M$ define

$d(x, M) = \inf \{d(x, y) : y \in M\}$ (1)

The function $d(\cdot, M) : X \to [0,\infty)$ is called the distance function associated to M. If there is m in M such that $d(x, m) = d(x, M)$, then m is called nearest point to x in M or projection of x in M. The set of nearest points of M to $x\in X$ is denoted by $P_M(x)$ and defined by

$P_M(x) = \{y\in M: d(x, y) = d(x, M)\}$.

The mapping $P_M : X \to M$ is called metric projection. If $P_M(x) \neq \emptyset$ for all $x \in X$, then M is called a proximinal set or a set of existence. If $P_M(x)$ contains at most one element for all $x \in X$, then M is called a set of uniqueness. If $P_M(x)$ contains a singleton set for all $x \in X$, then M is called a Chebyshev set.

We face with the following problem:

Problem 1.2

(a) When can we guarantee there is a nearest point in some target set?

(b) When there is a nearest point, under what condition can we guarantee its uniqueness?

It's not hard to see that a satisfactory answer to this question will require that we take M to be a closed set in X, for otherwise points in $\overline{M} \setminus M$ (the boundary of the set M) will not have nearest points. Indeed, which point in the interval [0, 1) is nearest to 1? Less obvious is that we typically need to impose additional requirements on M in order to insure the existence (and certainly the uniqueness) of nearest points.

The following well-known result asserts that if M is a complete convex set in an inner product space X, then each $x \in X$ has a unique element of best approximation in M.

Theorem 1.3. Let M be a nonempty complete convex set in an inner product space X. Then M is a Chebyshev set.

Theorem 1.4. Let M be a nonempty closed convex set in a uniformly convex Banach space X. Then M is a Chebyshev set.

Theorem 1.5 [13] Let M be a subset in a metric space X and $x \in X$. If M is compact, then

$\exists\ v \in M : d(x, v) = d(x, M)$.

Theorem 1.6. Let M be a convex set in a strictly convex Banach space X. Then for any point x in $X \setminus M$, there is at most one point in M that is nearest to x.

Indeed, if v, w and $v \neq w$ in a convex set are nearest points to u, then ½(v + w) would be closer to u (impossible!) which can be seen in the figure 1 below.

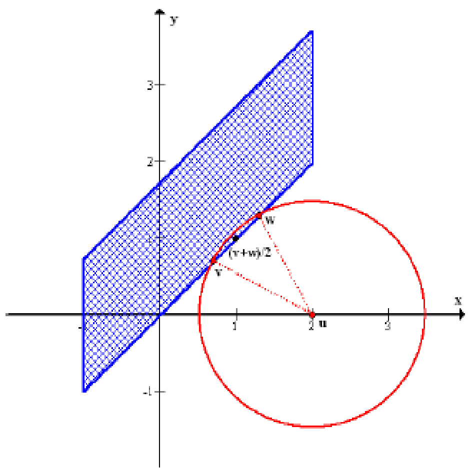

figure1.

Definition 1.7 [2] Let (A, B) be a pair of nonempty closed subsets of a metric space(X, d). The pair (A, B) is said to have P-property if

$d(x, u) = d(y, v) = d(A, B) \Rightarrow d(x, y) = d(u, v)$

where $x, y \in A$ and $u, v \in B$.

Definition 1.8. A metric space X is said to have P-property if every pair of nonempty closed sets in X has P-property.

Recently, Zhang et al. [12, 3, 5] introduced the new notion called weak P-property and showed that it is weaker than the P-property.

Definition 1.9 Let (A, B) be a pair of nonempty closed subsets of a metric space (X, d) with $A_0 \neq \phi$. The pair (A, B) is said to have weak P-property if

$d\ (x, u) = d(y, v) = \mathrm{dist}(A, B) \Rightarrow d(x, y) \leq d(u, v)$,

where $x, y \in A$ and $u, v \in B$.

Theorem 1.10 [13] Let M be a set in a metric space X and x in X. If M is compact and X has weak P-property, then

$\exists! v \in M : d(x, v) = d(x, M)$.

Proof. In **Theorem 1.5,** we have shown that $\exists\ v \in M : d(x, v) = d(x, M)$.

For the uniqueness, we observe that M and $\{x\}$ are closed sets in X. Since X has weak P-property, we have

$d(u, x) = d(v, x) = \mathrm{dist}(A, B) \Rightarrow \mathrm{d}(u, v) \leq d(x, x) = 0$.

taking $A = M$, $B = \{x\}$ with $u, v \in A$ in the definition of P-property. Thus,

$d(u, v) = 0 \Rightarrow u = v$. □

But in this paper, a new notion defined below is introduced and used for the existence and uniqueness of nearest point in a set.

Definition 1.11 Let (A, B) be a pair of nonempty closed subsets of a metric space (X, d) with $A_0 \neq \phi$. The pair (A, B) is said to have generalised weak P-property if

$\exists \alpha > 0$: $d(x, u) = d(y, v) = \mathrm{dist}(A, B) \Rightarrow d(x, y) \leq \alpha\ d(u, v)$

where $x, y \in A$ and $u, v \in B$.

Remark 1.12

(a) If $\alpha \in [0, 1]$, then generalised weak P-property implies weak P-property.

(b) If $\alpha \geq 1$, then weak P-property implies generalised weak P-property.

Example 1.13 Now we present an example which satisfies generalised weak P-property but not

P-property. Consider the set ℝ with the Euclidean metric and the subsets

$A = \{(0, 0)\}$ and $B = \{y = \sqrt{(1- x^2)}\}$.

Obviously,
$A_0 = \{(0, 0)\}$, $B_0 = \{(-1, 1), (1, 1)\}$ and $d(A, B) = \sqrt{2}$.
Furthermore,

$d((0, 0), (-1, 1)) = d((0, 0), (1, 1)) = \sqrt{2}$
however,
$0 = d((0, 0), (0,0)) < d((-1,1), (1, 1)) = \sqrt{2} d(A, B)$.

We see that the pair (A, B) satisfies the generalised weak *P*-property but not the *P*-property.

2. Uniqueness in the case of Compactness

Theorem 2.1. Let M be a set in a metric space X and $x \in X$. If M is compact and X has generalised weak *P*-property, then
$\exists!\, u \in M : ||x - u|| = d(x, M)$.
Proof. In Theorem 1.5, we have shown that

$\exists\, v \in M : d(x, v) = d(x, M)$.

For the uniqueness, we observe that M and $\{x\}$ are closed sets in X. Since X has generalised weak *P*-property, we have
$d(u, x) = d(v, x) = \text{dist}(A,B)\ ) \Rightarrow d(u, v) \leq \alpha\, d(x, x) = 0$.

taking $A = M$, $B = \{x\}$ with $u, v \in A$ in the definition of generalised weak *P*-property.

Thus, $d(u, v) = 0 \Rightarrow u = v$.

3. Uniqueness in the case of Weak Compactness

Lemma 3.1. Let M be a closed set in a Banach space X. If some minimizing sequence $\{z_n\} \subseteq M$ for $x \in X \setminus M$ has a weak cluster point z which lies in M then z is a nearest point to x in M.

Proof. By the weak lower semi-continuity of the norm we have

$d(x, M) \leq || x - z || \leq \liminf ||x - z_n|| \leq \lim || x - zn || = d(x, M)$.

so that z is a nearest point to x in M.

Definition 3.2. We say that M is boundedly weakly compact provided that $M \setminus B\,[0, r]$ is weakly compact for every $r > 0$.
Theorem 3.3. Let X be a Banach space. If M is non-empty and boundedly weakly compact and X has the generalised weak *P*-property, then M is Chebyshev.
Proof. Suppose that $x \in X \setminus M$ and let $\{z_n\}$ be a minimizing sequence in M for x. Then $\{z_n\}$ lies $M \setminus B[0,r]$ for some $r > 0$, and so has a weak cluster point z belonging to M. By Lemma 3.1, z is a nearest point to x.
For the uniqueness, we observe that M and $\{x\}$ are closed sets in X. Since X has generalised weak *P*-property, we have
$d(u, x) = d(v, x) = \text{dist}(A,B)\ ) \Rightarrow d(u, v) \leq \alpha\, d(x, x) = 0$.

taking $A = M$, $B = \{x\}$ with $u, v \in A$ in the definition of generalised weak *P*-property.

Thus, $d(u, v) = 0 \Rightarrow u = v$.

As a consequence, we have the following.

Theorem 3. 4. Let X be a reflexive Banach space. If X has generalised weak *P*-property, then a closed non-empty convex set in X is Chebyshev.

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